

# Vibration Of Finite String With Fixed Ends



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April 6, 2020

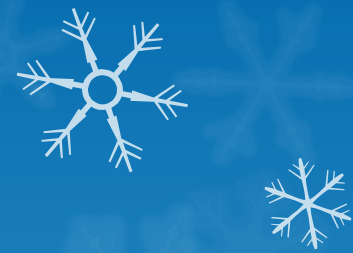
Suppose that vibration of the string of length  $L$  fixed at both sides (ends).

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \quad (1)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L \quad (2)$$

$$u_t(x, 0) = g(x), \quad 0 \leq x \leq L \quad (3)$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0 \quad (4)$$



We know that the solution of the wave equation  $u(x, t) = \phi(x + ct) + \varphi(x - ct)$

Use the initial conditions, we obtain

$$u(x, 0) = \phi(x) + \varphi(x) = f(x), \quad 0 \leq x \leq L \quad (5)$$

$$u_t(x, 0) = c [\phi'(x) - \varphi'(x)] = g(x), \quad 0 \leq x \leq L \quad (6)$$

Solving from equation (5) and (6), we get the value of  $\phi(v)$  and  $\varphi(w)$

$$\phi(v) = \frac{1}{2}f(v) + \frac{1}{2c} \int_0^v g(\alpha) d\alpha + \frac{k}{2}, \quad 0 \leq v \leq L \quad (7)$$

$$\varphi(w) = \frac{1}{2}f(w) - \frac{1}{2c} \int_0^w g(\alpha) d\alpha - \frac{k}{2}, \quad 0 \leq w \leq L \quad (8)$$



Therefore, the required solution

$$u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\alpha) d\alpha,$$

for,  $0 \leq x + ct \leq L$  and  $0 \leq x - ct \leq L$ . The solution is

determined by the initial data  $t \leq \frac{x}{c}$ , and  $t \leq \frac{L-x}{c}$ ,  $t \geq 0$ .

For larger times, applying the boundary conditions, we obtain

$$u(0, t) = \phi(ct) + \varphi(-ct) = 0, \quad t \geq 0, \quad (9)$$

$$u(L, t) = \phi(L + ct) + \varphi(L - ct) = 0, \quad t \geq 0, \quad (10)$$

Setting  $\beta = -ct$ , equation (9) reduces to

$$\varphi(\beta) = -\phi(-\beta), \quad \beta \leq 0, \quad (11)$$

Setting  $\beta = L + ct$ , equation (10) takes the form

$$\phi(\beta) = -\varphi(2L - \beta), \quad \beta \geq L, \quad (12)$$

If we take  $v = -w$ , then rewrite the equation (7) as

$$\phi(-w) = \frac{1}{2}f(-w) + \frac{1}{2c} \int_0^{-w} g(\alpha) d\alpha + \frac{k}{2}, \quad 0 \leq -w \leq L \quad (13)$$

Now, from equation (11) and (13), we obtain

$$\varphi(w) = -\frac{1}{2}f(-w) - \frac{1}{2c} \int_0^{-w} g(\alpha) d\alpha - \frac{k}{2}, \quad -L \leq w \leq 0 \quad (14)$$

We observe that the range of  $\varphi(w)$  is extended to  $-L \leq w \leq L$

If we taken  $\beta = v$  then equation (12) reduces to

$$\phi(v) = -\varphi(2L - v), \quad v \geq L, \quad (15)$$

Putting  $w = 2L - v$  in equation (8), we get

$$\varphi(2L - v) = \frac{1}{2}f(2L - v) - \frac{1}{2c} \int_0^{2L-v} g(\alpha) d\alpha - \frac{k}{2}, \quad 0 \leq 2L - v \leq L \quad (16)$$

Now from equation (15) and (16), we obtain

$$\phi(v) = -\frac{1}{2}f(2L - v) + \frac{1}{2c} \int_0^{2L-v} g(\alpha) d\alpha + \frac{k}{2}, \quad L \leq v \leq 2L \quad (17)$$

In above expression, the range of  $\phi(v)$  is extended to the interval

$L \leq v \leq 2L$ . Therefore, we obtain  $\phi(v)$ ,  $\forall v \geq 0$  and  $\phi(w)$ ,

$\forall w \leq L$ . The solution is determined  $\forall 0 \leq x \leq L$  and  $t \geq 0$ .

**Question 1:** Determine the solution of the following problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

$$u(x, 0) = \sin\left(\frac{\pi x}{L}\right), \quad 0 \leq x \leq L$$

$$u_t(x, 0) = 0, \quad 0 \leq x \leq L$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0$$



**Question 2:** Determine the solution of the following problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

$$u(x, 0) = f(x) \quad 0 \leq x \leq L$$

$$u_t(x, 0) = g(x), \quad 0 \leq x \leq L$$

$$u(0, t) = p(t), u(L, t) = q(t), t \geq 0$$



**Thank you**

