Lecture-X Programme: M. Sc. Physics

Liénard - Wiechert Potentials

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Contents

Liénard - Wiechert Potentials due to Point Charge

- **In this lecture we are going to calculate the (retarded) potentials, *V(*r*, t)* and A*(*r*, t)*, of a point charge *q* that is moving on a précised path-**

$$\mathbf{w}(t) \equiv \text{position of } q \text{ at time } t \qquad \text{-------------} \quad [1]$$

- **A easy reading of the formula from previous study is that-**

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau' \qquad \text{-------------} \quad [2]$$

❖ This suggests to you that the potential is merely

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} \qquad \text{-------------} \quad [3]$$

(the similar as in the static case, with the accepting that $\mathscr{r}$ is the distance to the *retarded* position of the charge). Although this is incorrect, for a very delicate motive: It is correct that for a point source the denominator $\mathscr{r}$ comes outside the integral (means integral is independent of $\mathscr{r}$), and remains the quantity-

$$\int \rho(\mathbf{r}', t_r)\, d\tau', \qquad \text{-------------} \quad [4]$$

that is *not* equal to the charge of the particle (and depends, through t_r, on the location of the point r).

❖ **To find the total charge of a arrangement, we must integrate ρ over the whole distribution at *one* moment of time, however here the retardation, $t_r = t - \mathscr{r}/c$, forces us determine ρ at different times for different parts of the arrangement.**

❖ **If the source is moving, this will provide a unclear image of the total charge. You might consider that this problem would vanish for point charges, but it doesn't.**

❖ In electrodynamics, devised as it is in terms of ***charge*** and ***current densities***, a point charge must be viewed as the limit of an extended charge, when the size goes to zero. And for an extended particle, no matter how small, the retardation in Eq. 4 tosss in a factor $(1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c)^{-1}$ where $\mathbf{v}$ is the velocity of the charge at the retarded time, now the equation 4 becomes as-

$$\int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{q}{1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c}. \quad \text{-------------} \quad [5]$$

- **Proof**- This is a merely geometrical consequence, and it can assist to tell the story in a less conceptual framework. We will not have observed it, for evident motives, however the reality is that a train coming towards you seems a slight longer than it actually is, since the light you get from the caboose left earlier than the light you get simultaneously from the engine, and at that earlier time the train was beyond away (Fig. 1).

❖ In the interval it receives light from the caboose to journey the extra distance L’ the train itself travels a distance L’ – L:

$$\frac{L'}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v/c}.$$ -------------- [6]

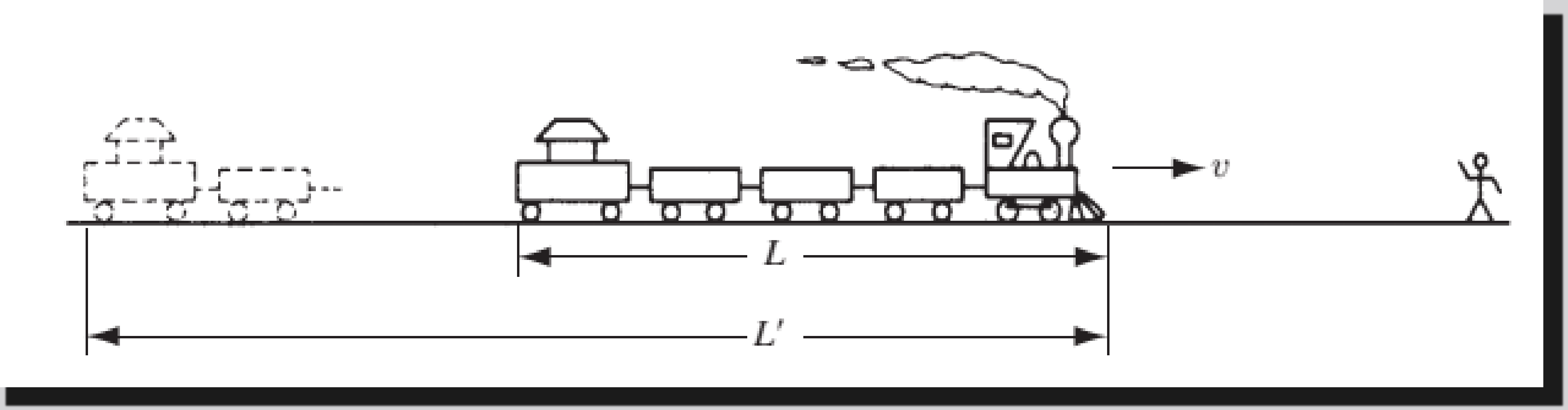

FIGURE 1:Representation of relative motion of a train and observer [*REF – 1]

❖ So coming near train appear longer, by a factor $(1 - v/c)^{-1}$. By contrast, a train going away from you seems *shorter*, by a factor $(1 + v/c)^{-1}$.

❖ In general, if the train's velocity makes an angle θ with your line of view, the additional distance light from the caboose must travel, is $L' \cos\theta$ (see Fig. 2). In the time $L' \cos\theta/c$, then, the train moves a distance $(L' - L)$:

$$\frac{L' \cos\theta}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v\cos\theta/c}. \qquad \text{-------------} \quad [7]$$

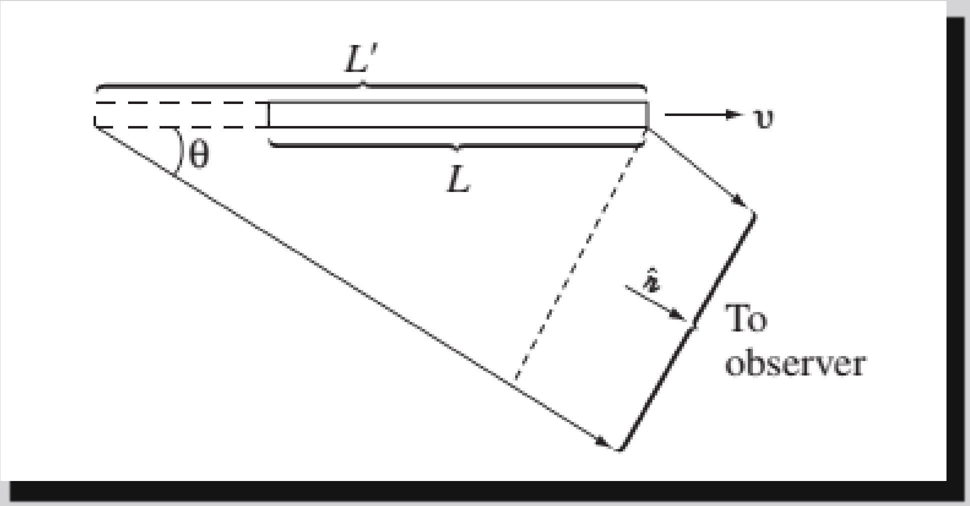

FIGURE 10.7: Representation for relative motion of objects [*REF – 1]

❖ It is obvious that this effect does *not* observed the dimensions perpendicular to the motion (the height and width of the train).

❖Never mind that the light from the distant side is late in reaching you (relative to light from the near side)—as there is no motion in this direction, they will still seem the similar distance apart.

❖The apparent *volume τ′* of the train, then, is linked to the *real* volume τ by-

$$\boxed{\tau' = \frac{\tau}{1 - \hat{\imath} \cdot \mathbf{v}/c},} \qquad \text{-------------- [8]}$$

where is $\hat{\imath}$ a unit vector from the train to the observer.

- In this case the link between moving trains and retarded potentials evades us, the point is this: when we do an integral of the type in Eq. 4, in which the integrand is calculated at the retarded time, the effective volume is customized by the factor in Eq. 8, just as the apparent volume of the train was. Since this adjustment factor builds no reference to the size of the particle, it is every bit as important for a point charge as for an extended charge.
- Finally, for a point charge the retarded time is obtained implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r). \qquad \text{-------------} \quad [9]$$

❖ The left side is the distance the "information" must travel, and *$(t - t_r)$* is the time it takes to make the journey (Fig. 3); $\mathscr{r}$ is the vector from the retarded site to the field point r:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r). \qquad \text{-------------- [10]}$$

It is significant to note that at most *one* point on the curve is "in communication" with r at any finite time *t*. For assume there were *two* such points, with retarded times t_1 and t_2:

$$\mathscr{r}_1 = c(t - t_1) \quad \text{and} \quad \mathscr{r}_2 = c(t - t_2). \qquad \text{-------------- [11]}$$

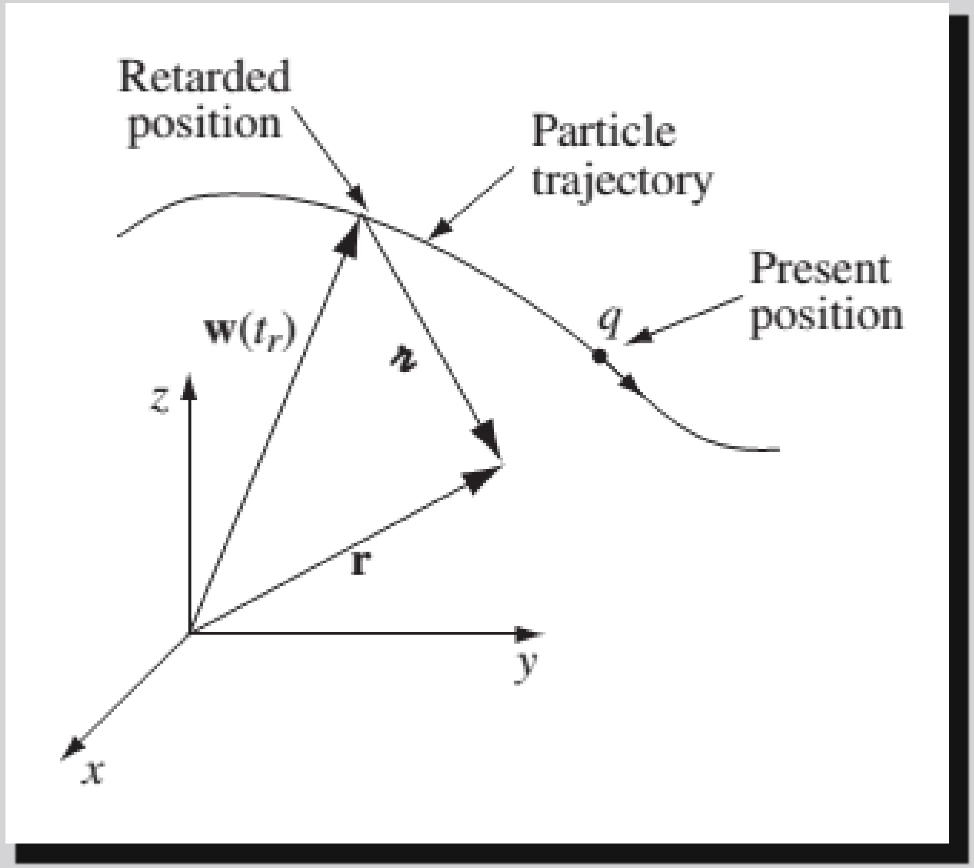

FIGURE 3: Representation of particle trajectory [*REF – 1]

❖ Then $\mathscr{r}_1 - \mathscr{r}_2 = c(t_2 - t_1)$, so the average speed of the particle in the direction of the point r would have to be *c*—and that's not including whatever velocity the charge might have in *other* directions.

❖ As no charged particle can move at the speed of light, it clears that only *one retarded point contributes to the potentials, at any known instant.*

❖Hence the potential defined in the eq. 2 is-

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \text{-------------} \quad [12]$$

here v is the velocity of the charge at the retarded time, and $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point r.

❖ In addition, as we know that the current density is product of charge density and velocity that is *ρ*v.

❖ Therefore the suitable expression of vector potential is-

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\rho(\mathbf{r}',t_r)\mathbf{v}(t_r)}{\mathscr{r}}\,d\tau' = \frac{\mu_0}{4\pi}\frac{\mathbf{v}}{\mathscr{r}}\int \rho(\mathbf{r}',t_r)\,d\tau',$$ --------- [13]

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t).$$ -------------- [14]

❖ Equations 12 and 14 are the well-known Liénard-Wiechert potentials for a moving point charge.

Numerical problems

1. Find the potentials of a point charge moving with constant velocity.
2. A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the x y plane, centered at the origin, and at time t = 0 the charge is at (a, 0), on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.
3. Determine the Liénard-Wiechert potentials for a charge in hyperbolic motion define by the equation- $$\mathbf{w}(t) = \sqrt{b^2 + (ct)^2}\,\hat{\mathbf{x}} \qquad (-\infty < t < \infty).$$

Assume the point r is on the x axis and to the right of the charge.

References:

1. *Introduction to Electrodynamics, David J. Griffiths*

2. *Elements of Electromagnetics, 2nd edition by M N O Sadiku.*

3. *Engineering Electromagnetics by W H Hayt and J A Buck.*

4. *Elements of Electromagnetic Theory & Electrodynamics, Satya Prakash*

- **For any query/ problem contact me on whatsapp group or mail on me**

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- **Next *** we will discuss the Retarded Potentials: Liénard - Wiechert Fields and Numerical problems related with this lecture.**

Thank you