Normal Spaces

Date: 12.05.2020

Def^n: Let X be a topological space in which singletons (and hence finite sets) are closed. Then X is said to be -

(i) 'regular' if $\forall\, x \in X$ and $\forall$ closed subset F in X such that $x \notin F$, $\exists$ disjoint open subsets U & V in X such that $x \in U$ and $F \subseteq V$.

(ii) 'normal' if $\forall$ disjoint closed subsets F_1, F_2 in X, $\exists$ disjoint open subsets G_1, G_2 in X such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$.

It is clear from the above definition that every normal space is obviously regular but converse of this is not necessarily true (it will be cleared from an example).

In some books, the above regular & normal spaces are also called as T_3 & T_4 respectively.

Example: (1) The Sorgenfrey line $\mathbb{R}_l$ is normal (and hence regular).

Sol^n: Note that basis for the space $\mathbb{R}_l$ is $\{[a,b) \mid a, b \in \mathbb{R}\}$. and $\mathbb{R}_l$ is finer than the usual space $\mathbb{R}$. Clearly $\forall a \in \mathbb{R}$, $\{a\}$ is closed in $\mathbb{R}_l$ as $\{a\} = \mathbb{R} - \{(-\infty, a) \cup (a, \infty)\}$. To show $\mathbb{R}_l$ normal, suppose that A & B are disjoint closed

sets in $\mathbb{R}_l$. Then $\forall a \in A$, as $a \notin B$ and B is closed $\Rightarrow \exists$ a basic element $[a, x_a)$ such that $[a, x_a) \cap B = \phi$. Similarly, $\forall b \in B$ $\exists$ a basic element $[b, x_b)$ such that $[b, x_b) \cap A = \phi$. Put $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$. Then clearly U and V are open in $\mathbb{R}_l$ with $A \subseteq U$ and $B \subseteq V$. It can also be easily verified that $U \cap V = \phi$ (use the fact that $A \cap B = \phi$), and so $\mathbb{R}_l$ is normal. #

Example: (2) The Sorgenfrey plan $\mathbb{R}_l^2$ is not normal, whereas it is regular.

Soln: This example clarifies that (i) normality is not productive and (ii) a regular space need not be normal.

It is obvious that $\mathbb{R}_l^2$ is regular (∵ $\mathbb{R}_l$ is regular, being it is normal, and regularity is productive [pl. check it], so is $\mathbb{R}_l^2$).

We will show that $\mathbb{R}_l^2$ is not normal by the method of contradiction.

If possible, suppose that $\mathbb{R}_l^2$ is normal

Let L (line, as given in fig. (i)) be the subspace of $\mathbb{R}_l^2$. Then clearly L is closed in $\mathbb{R}_l^2$ (as it is already closed in $\mathbb{R}^2$) and L has the discrete top. (as a subspace) [clear from

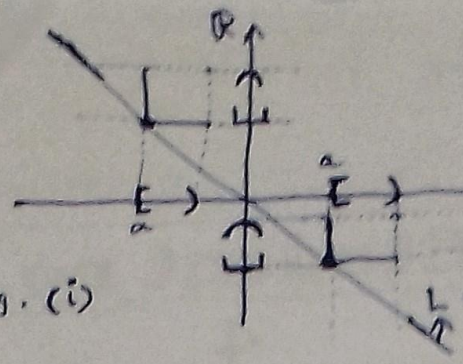

fig. (i) $L = \{(x, -x) \mid x \in \mathbb{R}\}$

the above fig.]. Hence every subset A of L, being closed in L, is also closed in $\mathbb{R}_l^2$. Thus, for every non-empty proper subset A of L, as $A \subsetneq L - A$ both are closed in $\mathbb{R}_l^2$, so by normality of $\mathbb{R}_l^2$, we have 'disjoint' open sets U_A and V_A in $\mathbb{R}_l^2$ such that $A \subseteq U_A$ and $L - A \subseteq V_A$.

Let $D = \mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2$. Then clearly, D is dense in $\mathbb{R}_l^2$.

Now define a map $f : \mathcal{P}(L) \to \mathcal{P}(D)$ as follows:

$$f(A) = \begin{cases} \phi & , \text{ if } A = \phi \\ D & , \text{ if } A = L \\ D \cap U_A & , \text{ if } \phi \neq A \neq L \end{cases} \quad , \forall A \subseteq L.$$

Then we claim that this f is one-one. The reason is as follows –

If A be proper non-empty subset of L. Then $f(A) = D \cap U_A$ is neither empty (∵ D is dense in $\mathbb{R}_l^2$) nor all of D. If we take another proper non-empty subset B of L,

then sets A and B must contain a point not in the other, so let, $x \in A$ but $x \notin B$. Then $x \in L - B$. But then, we notice that $x \in U_A \cap V_B$ ($\because A \subseteq U_A$ and $L - B \subseteq V_B$). Now $U_A \cap V_B$, being open in $\mathbb{R}^2_l$, must intersect with D, i.e., $(U_A \cap V_B) \cap D \neq \phi$. Thus $\exists\ y \in (U_A \cap V_B) \cap D$, but then $y \notin U_B$ ($\because U_A \cap U_B = \phi$), whereby we have $y \notin U_B \cap D$ but $y \in U_A \cap D$. Thus $U_A \cap D \neq U_B \cap D$. So $f(A) \neq f(B)$. Hence, f is one-one.

Now, we produce some injective (one-one) map from $\mathcal{P}(D) \to L$. as cardinality of D and cardinality of $\mathbb{Z}^+$ are the same, so it is enough to produce some injective map from $\mathcal{P}(\mathbb{Z}^+) \to \mathbb{R}$.

Now define $\varphi : \mathcal{P}(\mathbb{Z}^+) \to \mathbb{R}$ as follows:

$\varphi(S) = .a_1 a_2 \cdots\cdots$, where $a_i = 0$ if $i \in S$ and $a_i = 1$ if $i \notin S$, $\forall\ S \subseteq \mathbb{Z}^+$. Then obviously φ is one-one. Also note that cardinality of $\mathbb{R}$ and cardinality of L are the same.

Now the composite map $\mathcal{P}(L) \xrightarrow{f} \mathcal{P}(D) \xrightarrow{\varphi} L$ is one-one (here $\mathcal{P}(D) \equiv \mathcal{P}(\mathbb{Z}^+)$ and $L \equiv \mathbb{R}$, being of same cardinality).

Thus, we get a one-one map $\varphi \circ f : \mathcal{P}(L) \longrightarrow L$, which is a contradiction to the Cantor's theorem that

$\nexists$ any one-one map from $\mathcal{P}(X) \to X$, for any non-empty set X. Thus $\mathbb{R}_\ell^2$ cannot be normal. #

Proposition: If X and Y are top. space and Y is Hausdorff (T_2) and $f, g : X \to Y$ are continuous, then the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

Proof: Take $E = \{x \in X \mid f(x) = g(x)\}$. To show E is closed in X, we will show that $X - E$ is open in X (by showing that it is a nbd. of each of its points).

So, let $x \in X - E$. Thus $x \notin E$, so $f(x) \neq g(x)$ and Y is T_2, hence $\exists$ 'disjoint' open sets U & V in Y such that $f(x) \in U$ and $g(x) \in V$. But then we have $x \in f^{-1}(U) \cap g^{-1}(V)$. Note that $f^{-1}(U) \cap g^{-1}(V)$ is open in X ($\because f$ & g are cont.). We claim that $x \in f^{-1}(U) \cap g^{-1}(V) \subseteq X - E$, whereby it will follow that $X - E$ is a nbd. of x and proof will be completed. So, let $z \in f^{-1}(U) \cap g^{-1}(V)$. Then $f(z) \in U$ and $g(z) \in V$. As $U \cap V = \phi$, so, $f(z) \neq g(z) \Rightarrow z \notin E \Rightarrow z \in X - E$. #

References:

1. James R. Munkres, Topology, 2nd ed.